The Bouncing Ball Problem

A ball is dropped from a height of 4 meters. Each time the ball bounces, the height of its bounce is ½ of the previous height. We want to calculate the distance traveled by the ball before it stops.

We need to remember that on each *bounce* the ball travels UP and it falls back DOWN to the ground and that the UP *plus* the DOWN distances are part of the ball's total travel distance.

At first it appears that we need to use the equation for the sum of an infinite series,

$$S_\infty = \frac{a_0}{1-r}$$

where a_0 = the initial height and r = ½, but for the ball bouncing problem, we need to revise our thinking a bit, and will need to include an extra term.

Because we are dropping the ball from ABOVE the ground, the ball is *not* making a full UP-and-DOWN trip when it is dropped, but only traveling the DOWN part of the trip. It has not yet hit the ground and bounced.

In this problem, we assume that the ball's first bounce height will be

a_0 = (1/2)(4 meters) = 2 meters.

On the first bounce the ball travels UP 2 meters and DOWN 2 meters for a total distance of 4 meters traveled between the first time it comes in contact with the ground (start of first bounce) and the second time it comes in contact with the ground (before it begins its second bounce).

The second time the ball comes in contact with the ground it starts its second bounce. The height of the second bounce will be

(1/2)(1/2)(4 meters) = $(1/2)^2$(4 meters) = (1/2)(2 meters) = 1 meter.

On the second bounce the ball travels UP 1 meter and DOWN 1 meter for a total distance of 2 meters traveled between the second time it comes in contact with the ground (start of second bounce) and the third time it comes in contact with the ground (before it begins its third bounce).

The third time the ball comes in contact with the ground it starts its third bounce. The height of the third bounce will be

(1/2)(1/2)(1/2)(4 meters) = $(1/2)^3$(4 meters) = (1/2)(1 meter) = ½ meter.

On the second bounce the ball travels UP ½ meter and DOWN ½ meter for a total distance of 1 meter traveled between the third time it comes in contact with the ground (start of third bounce) and the fourth time it comes in contact with the ground (before it begins its fourth bounce).

We can continue this reasoning for each additional bounce resulting in:

$$S_\infty = 4 + 2(4)\left(\frac{1}{2}\right)^1 + 2(4)\left(\frac{1}{2}\right)^2 + 2(4)\left(\frac{1}{2}\right)^3 + \cdots$$

Factoring out 2(4), keeping mind that the factor of 2 accounts for the UP and DOWN travel distances:

$$S_\infty = 4 + 2(4)\left[\left(\frac{1}{2}\right)^1 + \left(\frac{1}{2}\right)^2 + \left(\frac{1}{2}\right)^3 + \cdots\right]$$

We need to add and subtract the term $2(4)\left(\frac{1}{2}\right)^0$ in the following step in order to use the summation formula below:

$$S_\infty = 4 - 2(4)\left(\frac{1}{2}\right)^0 + 2(4)\left[\left(\frac{1}{2}\right)^0 + \left(\frac{1}{2}\right)^1 + \left(\frac{1}{2}\right)^2 + \left(\frac{1}{2}\right)^3 + \cdots\right]$$

$$S_\infty = 4 - 2(4)\left(\frac{1}{2}\right)^0 + 2(4)\left[\left(\frac{1}{2}\right)^0 + \left(\frac{1}{2}\right)^1 + \left(\frac{1}{2}\right)^2 + \left(\frac{1}{2}\right)^3 + \cdots\right]$$

$$S_\infty = 4 - 8 + 2(4)\left[\left(\frac{1}{2}\right)^0 + \left(\frac{1}{2}\right)^1 + \left(\frac{1}{2}\right)^2 + \left(\frac{1}{2}\right)^3 + \cdots\right]$$

Simplifying the above with the summation formula:

$$2(4)\left[\left(\frac{1}{2}\right)^0 + \left(\frac{1}{2}\right)^1 + \left(\frac{1}{2}\right)^2 + \left(\frac{1}{2}\right)^3 + \cdots\right] = \sum_{k=0}^{\infty} 2(4)\left(\frac{1}{2}\right) = 2(4)\frac{1}{1-\frac{1}{2}}$$

Results in:

$$S_\infty = 4 - 8 + 2(4)\left[\frac{1}{1-\frac{1}{2}}\right]$$

$$S_\infty = 4 - 8 + 8\left[\frac{1}{\frac{1}{2}}\right]$$

$$S_\infty = -4 + \ 16 = 12$$

Our revised formula for the total travel distance for the ball is

$$S_\infty = -a_0 + 2a_0 \frac{1}{1-r} = -4 + (2)(4)\frac{1}{1-\frac{1}{2}} = -4 + \frac{8}{\frac{1}{2}} = -4 + 16 = 12 \text{ meters}$$

Example:

The ball is dropped from a height of 3 meters, and it bounces to ½ of the height of the previous bounce the total distance traveled by the ball is:

$$S_\infty = -a_0 + 2a_0 \frac{1}{1-r} = -3 + (2)(3)\frac{1}{1-\frac{1}{2}} = -3 + \frac{6}{\frac{1}{2}} = -3 + 12 = 9 \text{ meters}$$

The ball is dropped from a height of 2 meters, and bounces ¾ of the height of the previous bounce the total distance traveled by the ball is:

$$S_\infty = -a_0 + 2a_0 \frac{1}{1-r} = -2 + (2)(2)\frac{1}{1-\frac{3}{4}} = -2 + \frac{4}{\frac{1}{4}} = -2 + 16 = 14 \text{ meters}$$

Example:

The ball is dropped from a height of 10 meters, and bounces ¾ of the height of the previous bounce the total distance traveled by the ball is:

$$S_\infty = -a_0 + 2a_0 \frac{1}{1-r} = -10 + (2)(10)\frac{1}{1-\frac{3}{4}} = -10 + \frac{20}{\frac{1}{4}} = -10 + 80 = 70 \text{ meters}$$

Additional information:

https://demonstrations.wolfram.com/BounceTimeForABouncingBall/ for an example

http://www.sosmath.com/calculus/geoser/bounce/bounce.html for an interactive demo

https://www.youtube.com/watch?v=FN0wI_A9cqI YouTube video